Math 250 Briggs 2/e

7.2 Natural Log and Natural Exponential Functions - day 2

Objectives.

1) Integrate rational functions that result in antiderivatives containing natural logs

☆☆ 2) Integrate the remaining trig functions

$$\int \tan x \, dx$$

$$\int \cot x \, dx$$

$$\int \sec x \, dx$$

$$\int \csc x \, dx$$

* CAUTION * Objective #2 is not covered by our textbook but is agreed by all calc instructors to be taught in section 7.2.
Homework practice problems are provided on the supplemental homework problems handout.

3) Use division before integrating rational functions with degree of numerator greater than degree of denominator.

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Integrate:

× (3) $\int \frac{1}{4-3x}\,dx$

$u = 4-3x$
$du = -3\,dx$

$= -\frac{1}{3}\int \frac{1}{u}\,du$

$= -\frac{1}{3}\ln|u| + C$

$= \boxed{-\frac{1}{3}\ln|4-3x| + C}$

✓ (4) $\int \frac{x^2-2x}{x^3-3x^2}\,dx$

$u = x^3-3x^2$
$du = 3x^2-6x\,dx$
$du = 3(x^2-2x)\,dx$

$= \frac{1}{3}\int \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u| + C$

$= \boxed{\frac{1}{3}\ln|x^3-3x^2| + C}$

Not a ln

✓ (5) $\int \frac{x}{\sqrt{9-x^2}}\,dx$

$u = 9-x^2$
$du = -2x\,dx$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{2}\cdot 2u^{1/2} + C$

$= -1\sqrt{9-x^2} + C$

$= \boxed{-\sqrt{9-x^2} + C}$

Division needed ✓

(6) $\int \frac{2x^2+7x-3}{x-2}\,dx$

deg(numerator) > deg(denom)
⇒ use division
[long or synthetic]

2)	2	7	−3
		4	22
	2	11	19

$= \int \left(2x + 11 + \frac{19}{x-2}\right)dx$

$= \frac{2}{2}x^2 + 11x + 19\ln|x-2| + C$

$= \boxed{x^2 + 11x + 19\ln|x-2| + C}$

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Recall derivatives of trig functions:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

✓ 7 $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x}\, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int \frac{1}{u}\, du$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos x| + C}$$

X 8 $\int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x}\, dx$$

$u = \sin x$

$du = \cos x$

$$= \int \frac{1}{u}\, du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\sin x| + C}$$

Note: The signs of these two are unlike previous patterns!

X 9 $\int \sec x \, dx$

To get a derivative containing secant, need either sec or tan in denom, but neither by itself will quite work out, so use both.

$$= \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x \, dx$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$

✓ 10 $\int \csc x \, dx$

$$= \int \frac{\csc x(\csc x + \cot x)}{(\csc x + \cot x)}\, dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx$$

$u = \csc x + \cot x$

$du = -\csc x \cot x - \csc^2 x \, dx$

$$= -\int \frac{1}{u}\, du$$

$$= \boxed{-\ln|\csc x + \cot x| + C}$$

Suggestion: Do these four problems several times until the process and answers are ingrained in your memory.

Math 250 7.2 Supplemental Homework problems – Integrating Trig Functions

Find the indefinite integral:

1) $\int \cot\frac{\theta}{3}\,d\theta$

2) $\int \csc 2x\,dx$

3) $\int (\cos 3\theta - 1)\,d\theta$

4) $\int \frac{\cos t}{1+\sin t}\,dt$

5) $\int \frac{\sec x \tan x}{\sec x - 1}\,dx$

6) $\int (\sec 2x + \tan 2x)\,dx$

7) $\int \tan 5\theta\,d\theta$

8) $\int \sec\frac{x}{2}\,dx$

9) $\int \left(2 - \tan\frac{\theta}{4}\right)d\theta$

10) $\int \frac{\csc^2 t}{\cot t}\,dt$

Evaluate the definite integral

11) $\int_1^2 \frac{1-\cos\theta}{\theta - \sin\theta}\,d\theta$

12) $\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta$

13) $\int_{\pi/4}^{\pi/2} \csc x - \sin x\,dx$

14) $\int_{-\pi/4}^{\pi/4} \frac{\sin x - \cos^3 x}{\cos^2 x}\,dx$

Solutions:

1) $3\ln\left|\sin\dfrac{\theta}{3}\right|+C$

2) $-\dfrac{1}{2}\ln\left|\csc 2x+\cot 2x\right|+C$

3) $\dfrac{1}{3}\sin(3\theta)-\theta+C$

4) $\ln\left|1+\sin t\right|+C$

5) $\ln\left|\sec x-1\right|+C$

6) $\dfrac{1}{2}\ln\left|\sec 2x+\tan 2x\right|-\ln\left|\cos 2x\right|+C$

7) $-\dfrac{1}{5}\ln\left|\cos 5\theta\right|+C$

8) $2\ln\left|\sec\dfrac{x}{2}+\tan\dfrac{x}{2}+C\right|$

9) $2\theta+4\ln\left|\cos\dfrac{\theta}{4}\right|+C$

10) $-\ln\left|\cot t\right|+C$

11) $\ln\left|\dfrac{2-\sin 2}{1-\sin 1}\right|$

12) $-\cot 0.4+\csc 0.4-0.2+\cot 0.2-\csc 0.2+0.1$

13) $\ln\left(\sqrt{2}+1\right)-\dfrac{\sqrt{2}}{2}$

14) $-\sqrt{2}$

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× (11) $\int \frac{1}{x \ln x^3}\,dx$

Method 1

$u = \ln x^3$

$du = \frac{1}{x^3} \cdot 3x^2\,dx$

$du = \frac{3}{x}\,dx$

mult + div by 3 $= \frac{1}{3}\int \frac{3}{x \ln x^3}\,dx$

reorder $= \frac{1}{3}\int \frac{1}{\ln x^3} \cdot \frac{3}{x}\,dx$

rewrite w/ u $= \frac{1}{3}\int \frac{1}{u}\,du$

antidiff $= \frac{1}{3}\ln|u| + C$

subst back $= \boxed{\frac{1}{3}\ln|\ln x^3| + C_1}$

Method 2:

$= \int \frac{1}{3x \ln x}\,dx$

log property $\ln x^3 = 3\ln x$

$u = \ln x$

$du = \frac{1}{x}\,dx$

factor out $\frac{1}{3}$ $= \frac{1}{3}\int \frac{1}{\ln x} \cdot \frac{1}{x}\,dx$

rewrite w/ u $= \frac{1}{3}\int \frac{1}{u}\,du$

anti diff $= \frac{1}{3}\ln|u| + C$

subst back $= \boxed{\frac{1}{3}\ln|\ln x| + C_2}$

But why are these answers equivalent?

$\frac{1}{3}\ln|\ln x^3| + C_1$

$= \frac{1}{3}\ln|3 \cdot \ln x| + C_1$

$= \frac{1}{3}\ln(3 \cdot |\ln x|) + C_1$

$= \frac{1}{3}\ln 3 + \frac{1}{3}\ln|\ln x| + C_1$

$C_2 = C_1 + \frac{1}{3}\ln 3$

$= \frac{1}{3}\ln|\ln x| + C_2$

which is the same result.

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(12) $\displaystyle\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx$

Notice: degree of numerator > degree of denominator

$$\begin{array}{r|l} & x-3 \\ \hline x^2+3 & x^3-3x^2+4x-9 \\ & -x^3 \quad\quad \mp 3x \\ \hline & -3x^2+x-9 \\ & \pm 3x^2 \quad\quad \pm 9 \\ \hline & x \end{array}$$

can't use synthetic division because divisor is not linear

$= \displaystyle\int x-3+\frac{x}{x^2+3}\,dx$

rewrite using quotient and remainder

$= \displaystyle\int x-3\,dx + \int \frac{x}{x^2+3}\,dx$

$u=x^2+3$
$du=2x\,dx$

$= \frac{1}{2}x^2-3x+2\displaystyle\int \frac{du}{u}$

$= \frac{1}{2}x^2-3x+2\ln|u|+C$

$= \frac{1}{2}x^2-3x+2\ln|x^2+3|+C$

notice $x^2+3>0$ for all x So absolute values not needed.

$= \boxed{\frac{1}{2}x^2-3x+2\ln(x^2+3)+C}$

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(13) $\int 2 - \tan\frac{\theta}{4}\, d\theta$

$= \int 2\, d\theta - \int \tan\frac{\theta}{4}\, d\theta$ $\qquad u = \frac{\theta}{4}$, $du = \frac{1}{4} d\theta$

$= 2\theta - 4\int \tan u\, du$ $\qquad \leftarrow$ recall $\tan u = \frac{\sin u}{\cos u}$ where $w = \cos u$, $dw = -\sin u$

$= 2\theta + 4\ln|\cos u| + c$

$= \boxed{2\theta + 4\ln\left|\cos\frac{\theta}{4}\right| + c}$

(14) Solve the initial value problem

$\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1}$

$r(\pi) = 4$

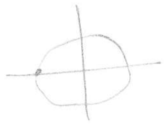
$\int dr = \int \frac{\sec^2 t}{\tan t + 1}\, dt$ $\qquad u = \tan t + 1$, $du = \sec^2 t\, dt$

$r(t) = \int \frac{du}{u}$

$r(t) = \ln|u| + c$

$r(t) = \ln|\tan t + 1| + c$

$r(\pi) = \ln|\tan\pi + 1| + c = 4$



$\ln|0 + 1| + c = 4$

$\ln(1) + c = 4$

$0 + c = 4$

$\boxed{r(t) = \ln|\tan t + 1| + 4}$

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(15) $\displaystyle\int \frac{x(x-2)}{(x-1)^3}\,dx$

$u = x-1 \qquad x = u+1$

$du = dx$

$$= \int \frac{(u+1)(u+1-2)}{u^3}\,du$$

$$= \int \frac{(u+1)(u-1)}{u^3}\,du$$

$$= \int \frac{u^2-1}{u^3}\,du$$

$$= \int \frac{u^2}{u^3} - \frac{1}{u^3}\,du$$

$$= \int \frac{1}{u}\,du - \int u^{-3}\,du$$

$$= \ln|u| - (-2)u^{-2} + c$$

$$= \boxed{\ln|x-1| + 2(x-1)^{-2} + c}$$

or $\boxed{\ln|x-1| + \dfrac{2}{(x-1)^2} + c}$

ugly change of variables ✗ Math 250

(16) $$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$$

$$u = \sqrt[3]{x} - 1 = x^{1/3} - 1$$

$$du = \frac{1}{3}x^{-2/3}\,dx = \frac{1}{3\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}}\,dx$$

$$du = \frac{\sqrt[3]{x}}{3x}\,dx$$ ← uh-oh. Need 3x.

$$u = x^{1/3} - 1$$
$$u + 1 = x^{1/3}$$
$$(u+1)^3 = x$$ ← Here's what we'll use for x.

$$= \int \frac{3x}{\sqrt[3]{x}-1} \cdot \frac{\sqrt[3]{x}\,dx}{3x}$$ ← rewrite with $\frac{3x}{3x}$

$$= \int \frac{3(u+1)^3}{u} \cdot du$$ ← Expand using Binomial Theorem, or FOIL and then multiply.

$$= \int \frac{3(u^3 + 3u^2 + 3u + 1)}{u}\,du$$

$$= 3\int \left(\frac{u^3}{u} + \frac{3u^2}{u} + \frac{3u}{u} + \frac{1}{u}\right)du$$ ← separate + simplify

$$= 3\int \left(u^2 + 3u + 3 + \frac{1}{u}\right)du$$

$$= 3\left[\frac{u^3}{3} + \frac{3}{2}u^2 + 3u + \ln|u|\right] + C_1$$ ↖ integrate all

$$= u^3 + \frac{9}{2}u^2 + 9u + 3\ln|u| + C_1$$

$$= (\sqrt[3]{x}-1)^3 + \frac{9}{2}(\sqrt[3]{x}-1)^2 + 9(\sqrt[3]{x}-1) + 3\ln\left|\sqrt[3]{x}-1\right| + C_1$$ ← subst back

$$= x - 3\sqrt[3]{x^2} + 3\sqrt[3]{x} - 1 + \frac{9}{2}\left(\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1\right) + 9\sqrt[3]{x} - 9 + 3\ln\left|\sqrt[3]{x}-1\right| + C_1$$ ← simplify

$$= x - 3\sqrt[3]{x^2} + 12\sqrt[3]{x} - 10 + \frac{9}{2}\sqrt[3]{x^2} - 9\sqrt[3]{x} + \frac{9}{2} + 3\ln\left|\sqrt[3]{x}-1\right| + C_1$$

$$= x + \frac{3}{2}\sqrt[3]{x^2} + 3\sqrt[3]{x} - \frac{11}{2} + 3\ln\left|\sqrt[3]{x}-1\right| + C_1$$

$$= \boxed{3\ln\left|\sqrt[3]{x}-1\right| + x + \frac{3}{2}x^{2/3} + 3x^{1/2} + C}$$

$$\left\{ C = C_1 - \frac{11}{2} \right\}$$
New constant

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(17) Find the average value of $f(x) = \sec \frac{\pi x}{6}$ on $[0, 2]$

average value $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$

$$= \frac{1}{2-0} \int_0^2 \sec \frac{\pi x}{6}\,dx$$

$u = \frac{\pi x}{6}$

$du = \frac{\pi}{6}\,dx$

$u(0) = \frac{\pi(0)}{6} = 0$

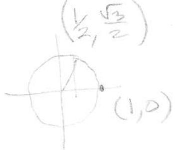
$u(2) = \frac{\pi(2)}{6} = \frac{\pi}{3}$

$$= \frac{1}{2} \cdot \frac{6}{\pi} \int_0^{\pi/3} \sec u\,du$$

$$= \frac{3}{\pi} \left(\ln|\sec u + \tan u| \right) \Big|_0^{\pi/3}$$

$$= \frac{3}{\pi} \left[\ln\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \ln(\sec 0 + \tan 0) \right]$$

$$= \frac{3}{\pi} \left[\ln(2 + \sqrt{3}) - \ln(1 + 0) \right]$$



$$= \frac{3}{\pi} \left[\ln(2+\sqrt{3}) - \ln 1 \right]$$

$$= \frac{3}{\pi} \left[\ln(2+\sqrt{3}) - 0 \right]$$

$$= \boxed{\frac{3}{\pi} \ln(2+\sqrt{3})} \approx 1.2576$$

GC will not converge
can't check that way.

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(18) Evaluate to nearest ten-thousandth.

$$\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2 \, d\theta$$

$u = 2\theta$
$du = 2\,d\theta$
$u(0.1) = 0.2$
$u(0.2) = 0.4$

$$= \frac{1}{2}\int_{0.2}^{0.4} (\csc u - \cot u)^2 \, du$$

$$= \frac{1}{2}\int_{0.2}^{0.4} \csc^2 u - 2\csc u \cot u + \cot^2 u \;\; du$$

already derivatives

NOT. Need trig identity

$$\frac{\sin^2 u}{\sin^2 u} + \frac{\cos^2 u}{\sin^2 u} = \frac{1}{\sin^2 u}$$

$$1 + \cot^2 u = \csc^2 x$$

$$\cot^2 u = \csc^2 u - 1$$

$$= \frac{1}{2}\int_{0.2}^{0.4} \csc^2 u - 2\csc u \cot u + \csc^2 u - 1 \;\; du$$

$$= \frac{1}{2}\int_{0.2}^{0.4} 2\csc^2 u - 2\csc u \cot u - 1 \;\; du$$

$$= \frac{1}{2}\left[-2\cot u + 2\csc u - u\right]\Bigg|_{0.2}^{0.4}$$

$$= \frac{1}{2}\left[(-2\cot 0.4 + 2\csc 0.4 - 0.4) - (-2\cot 0.2 + 2\csc 0.2 - 0.2)\right]$$

$$= \frac{1}{2}\Big[-2\cos(0.4)/\sin(0.4) + 2/\sin(0.4) - 0.4$$
$$+ 2\cos(0.2)/\sin(0.2) - 2/\sin(0.2) + 0.2\Big]$$

$$\approx \boxed{.0024}$$

fnInt $(Y_1, X, .1, .2) = .0024$ ✓